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Surveying in rotating systems II. Triangulation radius

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Abstract. It is shown that the radius measured by triangulation from a chord of a rotating system is identical to that obtained by using radar.

It is well known that the radar measurement of distance in an inertial frame gives an identical result to that obtained by laying off angles of sight from two points at a known distance apart (triangulation). Triangulation measurement of distance from a baseline which is rotating is already inherent in many measurements on the surface of the earth and is presumed to agree with radar to a high order of accuracy. It is also the technique applied to determine the distance of the nearer stars (parallax measurement) using the baseline provided by the earth's orbit around the sun. There is, however, no analysis to show that triangulation and radar measurements ever agree in rotating systems. This short paper derives the equation for the triangulation rotating radius and shows that it is identical to the radar rotating radius provided that the baseline has terminals on the same ring. It corroborates and strengthens the application of the formula for a rotating radius (Jennison 1964, Davies and Jennison 1975). It also gives support to the application of instantaneous Lorentz frames at vanishingly small regions in such systems (Ashworth and Jennison 1976).

Consider the rotating system of figure 1. The system is rotating at a constant angular velocity, ω, about the centre, O, according to observers in the inertial frame of the centre. We specify that the chord, AB, of the circle maintains the same proper length, L, irrespective of its velocity, i.e. it behaves as an invariant standard of length in its own local frame. When the angular velocity of rotation is zero an observer at A (or B) 'sees' the centre of rotation at an angle $\phi(=90°)$ with respect to the velocity vector at A (or B). The angles OAB and OBA, together with the length L, define the triangle ABO, and the radius, r, of the circle is given by AO or BO such that $r=\frac{1}{2}(\mathrm{AO}+\mathrm{BO})$. However, when the system is rotating with a constant angular velocity about O an observer at A (or B) 'sees' the centre at an angle ϕ' with respect to the velocity vector at A (or B). ϕ' is the aberration angle at A (or B) and the angles O′AB and O′BA, together with the length L, define the triangle ABO′ where O′ is the apparent centre of rotation. Using the angles and distances as defined in figure 1 we see that

$$\delta=\beta+\alpha, \qquad \epsilon=\alpha-\beta \qquad \text{and} \qquad \psi'=180-(\delta+\epsilon)=180-2\alpha=\psi.$$

Hence, the locus of O′ is the circumscribed circle to the triangle ABO.

According to the aberration formula

$$\cos\phi'=\frac{\cos\phi-v/c}{1-(v/c)\cos\phi},$$

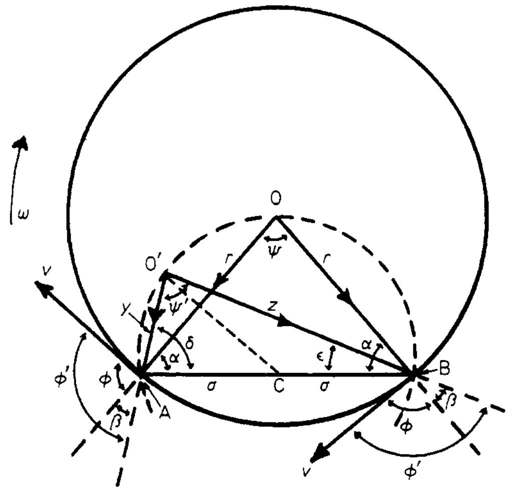

Figure 1. The aberrated direction to the centre seen from the terminals A and B of a chord of a synchronously rotating system. The off-set position of the centre in the diagram is simply a result of constructing rectilinear ray paths. Following Jennison (1963, 1964), the ray paths may be drawn as circular arcs of radius $\frac{1}{2}c/\omega' = \frac{1}{2}(c/\omega)(1-v^2/c^2)^{1/2}$, whence the centre is restored to the symmetrical position whilst the correct aberration angles remain at the ends of the chord.

where v is the velocity of rotation of A and B. Since $\phi = 90°$, $\cos\phi' = -v/c$ and $\sin\phi' = (1-v^2/c^2)^{1/2}$. Hence, since $\beta = \phi' - 90°$,

$$\cos\beta = (1-v^2/c^2)^{1/2} \qquad \text{and} \qquad \sin\beta = v/c. \tag{1}$$

But:

$$y = \frac{L\sin\epsilon}{\sin\psi} \qquad \text{and} \qquad z = \frac{L\sin\delta}{\sin\psi},$$

therefore

$$y = \frac{L}{\sin 2\alpha}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

and

$$z = \frac{L}{\sin 2\alpha}(\sin\beta\cos\alpha + \cos\beta\sin\alpha).$$

As a result of their observations, therefore, the rotating observers at A and B might be expected to conclude that, since $z > y$, the apparent centre O′ is nearer to A than to B. In this eventuality it is not obvious what should be taken as the radius obtained by parallax by the rotating observers. One way of overcoming this difficulty is to consider an infinitesimal parallax experiment in which the baseline AB is of length $L = 2\sigma$. The two observers may then use their observations to calculate O′C where C is the mid-point of AB, and then define the parallax radius to be given by $r' = \lim_{\sigma\to 0} \mathrm{O'C}$.

From the geometry of figure 1 it is readily shown that

$$\mathrm{O'C} = \sigma(1 - 2\cos 2\alpha\cos 2\beta + \cos^2 2\alpha)^{1/2}/\sin 2\alpha$$

whence, from equation (1) and the fact that $\cos\alpha = \sigma/r$ and $\sin\alpha = (r^2-\sigma^2)^{1/2}/r$, we obtain

$$O'C = \tfrac{1}{2}r[1-2(2\sigma^2/r^2-1)(1-2v^2/c^2)+(2\sigma^2/r^2-1)^2]^{1/2}(1-\sigma^2/r^2)^{-1/2}$$

thus giving

$$r' = r(1-v^2/c^2)^{1/2}. \tag{2}$$

The radius given by equation 2 is the same as that corresponding to radar measurements (Jennison 1964, Davies and Jennison 1975, Ashworth and Jennison 1976).

An alternative way of overcoming the above difficulty is to assume that the rotating observers at A and B realize that the off-set position of the apparent centre, O′, is simply a result of constructing rectilinear ray paths beyond the local Lorentz frames in which the angles are measured. If, following Jennison (1963), the ray paths are taken to be circular arcs of equal length then the centre is restored to the symmetrical position whilst the correct aberration angles remain at the ends of the chord. O′ in figure 1 is simply a 'virtual' centre resulting from the diagrammatic extrapolation. If we now, by analogy with the static case, define $r' = \frac{1}{2}(AO' + BO') = \frac{1}{2}(y+z)$ to be the radius of the rotating system according to observers rotating at an inertial radial distance r from the centre, then,

$$r' = \sigma \cos\beta/\cos\alpha$$

giving

$$r' = r(1-v^2/c^2)^{1/2},$$

as before. In this case, since β cannot exceed 90°, y and z must always be positive, hence

$$\sigma \leqslant r(1-v^2/c^2)^{1/2},$$

i.e.

$$\sigma \leqslant r'. \tag{3}$$

This simply means that the length of the baseline must never exceed the diameter corresponding to twice the radius measured by the rotating observer and therefore follows from the elementary restriction that the terminals of the baseline lie on the same ring.

An interesting result of this analysis is that there may be a limiting size to a measuring rod which is spun up to a particular velocity. The effect is most marked for a rod which is close to the centre. If, for example, a measuring rod is spun about its own centre, equation (3) shows that this limit is reached at zero angular velocity and the rod can no longer remain as a valid standard of length. This phenomenon is at the root of the Ehrenfest paradox.

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